Reflection and Refraction of Gaussian Light Beams at Tilted Ellipsoidal Surfaces

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This paper deals with the reflection and refraction of a gaussian laser beam at a curved interface between media of different refractive indices. The analysis extends beyond the usual case of normal incidence at spherical surfaces to include arbitrary angles of incidence and interfaces of ellipsoidal shape. By matching the transverse variations of optical phase at the interface, equations for the spot sizes and wavefront radii of the beams are obtained. These results have been converted to ray matrix form, which is particularly convenient for analyzing thick lenses or systems of several elements. With these matrices, one can readily design and evaluate optical systems containing such astigmatic elements as tilted spherical or cylindrical lenses and mirrors.

Introduction

It is sometimes necessary to evaluate the reflection and/or the refraction of a gaussian laser beam at a spherical or, in general, an ellipsoidal dielectric interface which is tilted from the optical axis of propagation. Examples include the use of spherical mirrors and lenses at nonnormal incidence; the estimation of errors caused by misoriented optical elements in an optical system; the calculation of astigmatism caused by plane or slightly curved surfaces placed inside optical resonators, especially at Brewster's angle; and the design of ring lasers and other optical systems containing elements with cylindrical or ellipsoidal surfaces. In connection with a problem of this type, we have extended the analysis of reflection and refraction at a dielectric interface beyond the well-known results for near-normal incidence and/or spherical surfaces, to include ellipsoidal surfaces and arbitrary angle of incidence.

Two limiting assumptions are made in the analysis. The first is that the beam diameter is always small compared with both the radius of curvature of the optical wavefronts and the radius of curvature of the optical surface. Thus, we consider, at most, quadratic variation of the wavefront phase and amplitude along the transverse coordinates.

The second assumption is that for any ellipsoidal surface (other than spherical or planar), one of the principle axes of the ellipsoidal surface must lie in the plane of incidence. To put this another way, for an ellipsoidal surface the plane of incidence of the light beam is restricted to one of the planes of symmetry of the optical element; or, the rotation of any tilted ellipsoidal element in an optical system is restricted to be about one of its two principal axes. With this assumption, our analysis does not cover all possible cases, but it probably does cover the majority of cases in which ellipsoidal (for example, cylindrical) elements are introduced deliberately into an optical system in order to achieve (or to compensate for) a particular astigmatic effect.

Development of the Analysis

The analysis is carried out by requiring that the transverse phase and amplitude variations of incident, reflected, and refracted waves must match exactly along the ellipsoidal boundary between two media of different refractive indices n_1 and n_2. Our analytical results can also be derived using ray-tracing arguments. The analysis based on phase matching at the boundary is presented here because this approach seems easier to present, using a minimum of illustrations, than the ray-tracing derivation.

Consider the geometry shown in Fig. 1. Separate coordinate systems for describing the incident, refracted, and reflected waves are denoted by (x_1,y_1,z_1), (x_2,y_2,z_2), and (x_3,y_3,z_3), respectively. The z axis in each case points along the propagation direction of the center of the beam; the x axes all lie in the plane of incidence; and the y axes are all normal to the plane of incidence in the direction necessary to create right-handed coordinate systems. The origins of all three coordinate systems are taken at the point of incidence on the interface.

The interface itself is represented by an ellipsoid in an (X,Y,Z) coordinate system. The angle of incidence

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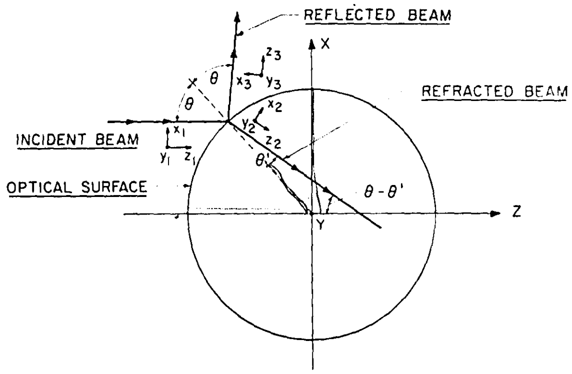

Fig. 1. Coordinate systems for surface and for incident, reflected, and refracted beams.

on the interface from the medium of index n_1 is Θ, and the refracted angle in the medium of index n_2 is Θ'. Because of the principal axis restriction introduced above, we may consider the interface to be part of an ellipsoid of revolution about the Y axis, with the X–Z plane corresponding to the plane of incidence, as shown in Fig. 2. If the semiaxes of the ellipse have lengths A and B, as shown in the figure, then the radius of curvature of the surface in the X–Z or tangential plane is $R_T = A$, and the radius of curvature in the orthogonal or sagittal plane at the point of intersection is $R_S = B^2/A$. Note that positive values of R_T and R_S correspond to a convex surface facing the incident beam.

The complex wave amplitude[1] for the fields in the three beams may be written in the form:

$$\tilde{u}_i(x_i,y_i,z_i) = \tilde{A}_i \exp(-j\phi_i[x_i,y_i,z_i]), \quad (i = 1,2,3), \tag{1}$$

where the complex phase ϕ_1 is given by

$$\phi_i(x_i,y_i,z_i) = k_i z_i + \frac{k_i}{2}\left(\frac{x_i^2}{q_{Ti}} + \frac{y_i^2}{q_{Si}}\right), \tag{2}$$

with the identifications: $i = 1 \leftrightarrow$ incident beam, $i = 2 \leftrightarrow$ refracted beam, $i = 3 \leftrightarrow$ reflected beam; T = tangential plane; S = sagittal plane:

$$k_1 = 2\pi n_1/\lambda_0 = k_3, \quad k_2 = 2\pi n_2/\lambda_0, \tag{3}$$
$$1/q_i \equiv 1/R_i - j(\lambda_0/\pi n_i w_i^2).$$

We are using the complex gaussian beam notation, in which R_i represents the wavefront radius of curvature (positive for a diverging beam); w_i is the beam radius to the $1/e$ field strength point; and the complex curvature q_i combines the wavefront radius and beam radius in the manner indicated. The phase ϕ_i is then also a complex quantity, combining both conventional phase information and amplitude information.

The ellipsoidal surface is represented in the (X,Y,Z) coordinate system by

$$(X^2 + Z^2)/A^2 + Y^2/B^2 = 1. \tag{4}$$

If we transform from this coordinate system to the incident wave coordinates by using the relations:

$$X = x_1 + A\sin\Theta,$$
$$Y = y_1, \tag{5}$$
$$Z = z_1 - A\sin\Theta,$$

then the ellipsoidal surface is described exactly by

$$(x_1 + A\sin\Theta)^2 + (Ay_1/B)^2 + (z_1 - A\cos\Theta)^2 = A^2. \tag{6}$$

A solution for z_1 on the interface, accurate to second order in the transverse variables x_1 and y_1, is

$$z_1 \approx x_1\tan\Theta + \frac{x_1^2}{2R_T\cos^3\Theta} + \frac{y_1^2}{2R_S\cos\Theta}, \tag{7}$$

where we have eliminated the ellipsoidal parameters A and B in favor of the local tangential and sagittal radii R_T and R_S. We may now write the complex phase of the incident wave on the interface surface in terms of x_1 and y_1 only, in the form:

$$\phi_1(x_1,y_1) = k_1x_1\tan\Theta + \frac{k_1x_1^2}{2}\left(\frac{1}{q_{T1}} + \frac{1}{R_T\cos^3\Theta}\right) + \frac{k_1y_1^2}{2}\left(\frac{1}{q_{S1}} + \frac{1}{R_S\cos\Theta}\right). \tag{8}$$

We will obtain the desired relationships between various q parameters by requiring that the complex phase variation of the incident, refracted, and reflected waves must match exactly on the interface, i.e.,

$$\phi_1(x_1,y_1) = \varphi_2(x_1,y_1) = \varphi_3(x_1,y_1). \tag{9}$$

Let us consider the reflected wave first, since it is slightly simpler.

Reflected Wave

For the reflected wave, we use the coordinate transformation:

$$x_3 = -x_1\cos2\Theta - z_1\sin2\Theta,$$
$$y_3 = y_1, \tag{10}$$
$$z_3 = x_1\sin2\Theta - z_1\cos2\Theta.$$

Substituting this together with Eq. (7) into $\phi_3(x_3,y_3,z_3)$ yields the result:

$$\varphi_3(x_1,y_1) = k_1x_1\tan\Theta + \frac{k_1x_1^2}{2}\left(\frac{1}{q_{T3}} + \frac{1 - 2\cos^2\Theta}{R_T\cos^3\Theta}\right) + \frac{k_1y_1^2}{2}\left(\frac{1}{q_{S3}} + \frac{1 - 2\cos^2\Theta}{R_S\cos\Theta}\right). \tag{11}$$

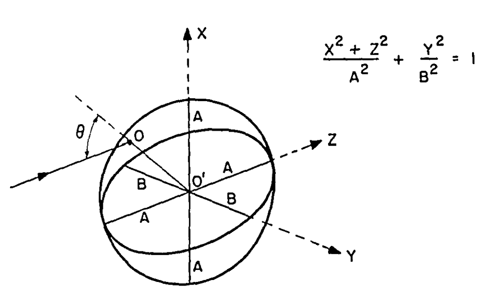

Fig. 2. Definition of ellipsoidal surface expanded about point of incidence O. Center of ellipsoid is at O', with semiaxes A and B.

Table I. Ray Matrix Elements

Matrix elements		Reflection	Refraction	Refraction at Brewster angle
A	Tangential	1	$\frac{(n_r^2 - \sin^2\Theta)^{\frac{1}{2}}}{n_r \cos\Theta}$	n_r
	Sagittal	1	1	1
B	Tangential or sagittal	0	0	0
C	Tangential	$2/R_T \cos\Theta$	$\frac{\cos\Theta - (n_r^2 - \sin^2\Theta)^{\frac{1}{2}}}{R_T \cos\Theta (n_r^2 - \sin^2\Theta)^{\frac{1}{2}}}$	$\frac{(1 - n_r^2)(n_r^2 + 1)^{\frac{1}{2}}}{R_T n_r}$
	Sagittal	$2 \cos\Theta/R_S$	$\frac{\cos\Theta - (n_r^2 - \sin^2\Theta)^{\frac{1}{2}}}{R_S n_r}$	$\frac{1 - n_r^2}{R_S n_r (n_r^2 + 1)^{\frac{1}{2}}}$
D	Tangential	1	$\frac{\cos\Theta}{(n_r^2 - \sin^2\Theta)^{\frac{1}{2}}}$	$1/n_r^2$
	Sagittal	1	$1/n_r$	$1/n_r$

Equating coefficients of x_1^2 and y_1^2 between Eqs. (8) and (11) yields the basic tangential and sagittal relations for reflection at the surface:

$$\frac{1}{q_{T3}} = \frac{1}{q_{T1}} + \frac{2}{R_T \cos\Theta} \text{ (tangential plane)},$$
$$\frac{1}{q_{S3}} = \frac{1}{q_{S1}} + \frac{2\cos\Theta}{R_S} \text{ (sagittal plane)}. \tag{12}$$

These relations imply the following relations between the spot sizes and wavefront radii in the tangential and sagittal planes:

$$\begin{aligned} w_{T3} &= w_{T1}, \\ 1/R_{T3} &= 1/R_{T1} + 2/(R_T \cos\Theta), \\ w_{S3} &= w_{S1}, \\ 1/R_{S3} &= 1/R_{S1} + 2\cos\Theta/R_S. \end{aligned} \tag{13}$$

For a spherical surface, these reduce to the results previously given by Collins.[2]

Refracted Wave

We obtain the refracted wave relations similarly, making use of the coordinate transformation:

$$\begin{aligned} x_2 &= x_1 \cos(\Theta - \Theta') + z_1 \sin(\Theta - \Theta'), \\ y_2 &= y_1, \\ z_2 &= -x_1 \sin(\Theta - \Theta') + z_1 \cos(\Theta - \Theta'). \end{aligned} \tag{14}$$

By expressing $\phi_2(x_2,y_2,z_2)$ in the (x_1,y_1,z_1) system, with z_1 eliminated by use of Eq. (7), we have on the interface:

$$\varphi_2(x_1,y_1) = k_2x_1[\tan\Theta \cos(\Theta - \Theta') - \sin(\Theta - \Theta')]$$
$$+ \frac{k_2x_1^2}{2}\left[\frac{\cos^2\Theta'}{\cos^2\Theta}\frac{1}{q_{T2}} + \frac{\cos(\Theta - \Theta')}{R_T \cos^3\Theta}\right] + \frac{k_2y_1^2}{2}\left[\frac{1}{q_{S2}} + \frac{\cos(\Theta - \Theta')}{R_S \cos\Theta}\right]. \tag{15}$$

By again equating coefficients of x_1^2 and y_1^2 in Eqs. (8) and (15), we obtain the results:

$$\frac{1}{q_{T1}} = \frac{\cos^2\Theta'}{\cos^2\Theta}\frac{n_r}{q_{T2}} + \frac{n_r \cos(\Theta - \Theta') - 1}{R_T \cos^3\Theta}, \tag{16}$$

$$\frac{1}{q_{S1}} = \frac{n_r}{q_{S2}} + \frac{n_r \cos(\Theta - \Theta') - 1}{R_S \cos\Theta}, \tag{17}$$

where we have used the convenient abbrevation: $n_r = n_2/n_1 = k_2/k_1$. By using Snell's law to eliminate Θ', we find that these results may also be written:

$$\frac{1}{q_{T2}} = \left(\frac{n_r \cos^2\Theta}{n_r^2 - \sin^2\Theta}\right)\frac{1}{q_{T1}} + \frac{n_r[\cos\Theta - (n_r^2 - \sin^2\Theta)^{\frac{1}{2}}]}{R_T[n_r^2 - \sin^2\Theta]},$$
$$\frac{1}{q_{S2}} = \frac{1}{n_r}\frac{1}{q_{S1}} + \frac{[\cos\Theta - (n_r^2 - \sin^2\Theta)^{\frac{1}{2}}]}{n_r R_S}. \tag{18}$$

The wavefront radius and spot size relations for the refracted wave are then given by

$$w_{T2} = \frac{(n_r^2 - \sin^2\Theta)^{\frac{1}{2}}}{n_r \cos\Theta} w_{T1},$$
$$\frac{1}{R_{T2}} = \left(\frac{n_r \cos^2\Theta}{n_r^2 - \sin^2\Theta}\right)\frac{1}{R_{T1}} + \frac{n_r[\cos\Theta - (n_r^2 - \sin^2\Theta)^{\frac{1}{2}}]}{R_T[n_r^2 - \sin^2\Theta]},$$
$$w_{S2} = w_{S1},$$
$$\frac{1}{R_{S2}} = \frac{1}{n_r}\frac{1}{R_{S1}} + \frac{\cos\Theta - (n_r^2 - \sin^2\Theta)^{\frac{1}{2}}}{n_r R_S}. \tag{19}$$

These results may also be obtained by conventional ray tracing.

Ray Matrix Results

It is also convenient to have the above results in the form of ray transfer matrices. We can readily transform to matrix notation by arbitrarily constructing a ray vector for the ith beam, in terms of the previously defined quantities w and R, of the form:

$$\text{Ray vector} = \begin{bmatrix} w_i \\ w_i/R_i \end{bmatrix}. \tag{20}$$

Following conventional ray vector notation, we have made w_1 in Eq. (20) the ray height, which is analogous to the spot size, and we have chosen w_i/R_i to be the ray slope, which is consistent with previous definitions since we are considering only slopes of magnitude much less than unity. We seek an A, B, C, D matrix of the form:

$$\begin{bmatrix} w_i \\ w_i/R_i \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} w_i \\ w_i/R_i \end{bmatrix} \tag{21}$$

to describe the ray transformations that occur upon reflection or refraction at the curved interface. This procedure is carried out for the tangential and sagittal planes. Writing out the individual relations in Eq. (21) and making use of our previous results, Eqs. (13) and (19), leads to the tangential and sagittal matrix elements for reflection and refraction summarized in Table I. Since the refraction matrix elements for Brewster angle incidence will be of particular interest in some applications, the elements for this case have been listed separately. A helpful check for carrying out numerical refraction calculations is the fact that the determinant of these matrices is always $n_1/n_2 = n_r^{-1}$ for refraction from a medium of index n_1 to a medium of index n_2.

Using the matrix form for the refraction relations, it is particularly easy to characterize the effect of a general second order, astigmatic thick lens, by simply cascading the appropriate surface matrices for the two lens surfaces on either side of the usual translation matrix:

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix},$$

where d is the distance traveled by the beam in refracting through the element.

Conclusion

The reflection and refraction of a gaussian laser beam at a tilted ellipsoidal interface between different dielectric media has been analyzed by matching the transverse variations of the fields at the interface. Carrying out the matching to second order yields relations between the q parameters, or between the spot sizes and wavefront radii, for the reflected and refracted waves. These results may also be transformed into the ray matrices characterizing refraction and reflection at the interface. Using these relations, a variety of practical problems involving off-axis lenses and mirrors, ellipsoidal optical elements, and various other astigmatic optical systems may be readily solved.

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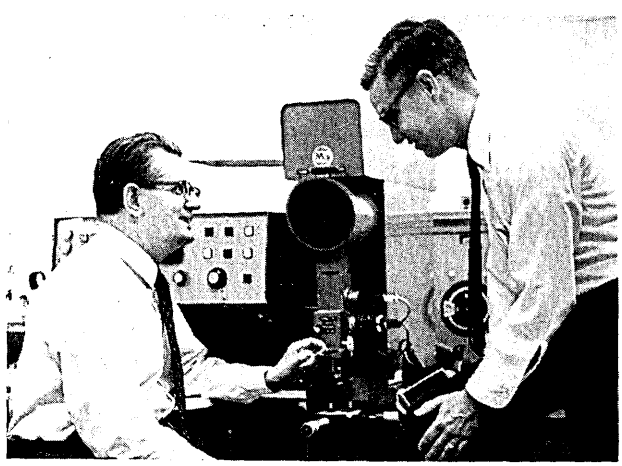
References

1. H. Kogelnik and T. Li, Appl. Opt. **5**, 1550 (1966).
2. S. A. Collins, Jr., Appl. Opt. **3**, 1263 (1964).



Guenter H. Schwuttke (left) and John M. Fairfield, of the IBM Components Division development laboratory here, discuss a diode they made by using light from a solid state laser. The quality of the experimental diode was comparable to that of devices made by conventional diffusion processes (see the December issue of *Solid-State Electronics*). The work was sponsored in part by the Air Force Cambridge Research Laboratories.