Capacitively Coupled Qubits

I. DERIVATION OF HAMILTONIAN

The Hamiltonian of two capacitively coupled qubits is

$$H = H^1 \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes H^2 + g\sigma_{z1} \otimes \sigma_{z2} \qquad \text{(Equation 1)}$$

where the single qubit Hamiltonians are of the form:

$$H^{i,(\mathrm{LR})} = \begin{pmatrix} \epsilon_i/2 & \Delta_i \\ \Delta_i & -\epsilon_i/2 \end{pmatrix} \qquad \text{(Equation 2)}$$

in the (LR) basis. The (LR) basis is the *Left-Right basis*, which is comprised of an electron occupying the left dot and an electron occupying the right dot.

We will refer to ϵ_i and Δ_i as the *detuning* and *tunnel coupling* of qubit i, respectively.

From this, we can write down the full Hamiltonian explicitly:

$$H_0^{(\mathrm{LR})} = \begin{pmatrix} \frac{1}{2}(\epsilon_1 + \epsilon_2) + g & \Delta_2 & \Delta_1 & 0 \\ \Delta_2 & \frac{1}{2}(\epsilon_1 - \epsilon_2) - g & 0 & \Delta_1 \\ \Delta_1 & 0 & \frac{1}{2}(-\epsilon_1 + \epsilon_2) - g & \Delta_2 \\ 0 & \Delta_1 & \Delta_2 & \frac{1}{2}(-\epsilon_1 - \epsilon_2) + g \end{pmatrix} \qquad \text{(Equation 3)}$$

II. GENERAL PERTURBATIONS IN THE ROTATING FRAME

Because we are imagining moving around our parameter space adiabatically, it makes sense for us to identify our logical states as the energy states of the Hamiltonian. So for the rest of this section, we will *not* use the (LR) basis, instead using the energy basis:

$$H_0 = \begin{pmatrix} E_{11} & 0 & 0 & 0 \\ 0 & E_{10} & 0 & 0 \\ 0 & 0 & E_{01} & 0 \\ 0 & 0 & 0 & E_{00} \end{pmatrix} \qquad \text{(Equation 4)}$$

We may also wish to add some perturbation to this Hamiltonian, such that

$$H = H_0 + H_1 \qquad \text{(Equation 5)}$$

where H_0 is given above and our perturbation is very general:

$$H_1 = \begin{pmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{pmatrix} \cos \omega t \qquad \text{(Equation 6)}$$

We wish to view our Hamiltonian H in the rotating frame.

The time dependent Schrödinger equation is

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H|\psi\rangle \qquad \text{(Equation 7)}$$

If we define

$$|\psi_R\rangle = U_R|\psi\rangle \qquad \text{(Equation 8)}$$

then

$$i\hbar\frac{\partial}{\partial t}|\psi_R\rangle = i\hbar\frac{\partial}{\partial t}U_R|\psi\rangle \qquad \text{(Equation 9)}$$

$$= i\hbar\left(\frac{\partial}{\partial t}U_R\right)|\psi\rangle + i\hbar U_R\left(\frac{\partial}{\partial t}|\psi\rangle\right) \qquad \text{(Equation 10)}$$

$$= i\hbar\left(\frac{\partial}{\partial t}U_R\right)|\psi\rangle + U_R H|\psi\rangle \qquad \text{(Equation 11)}$$

$$= i\hbar\left(\frac{\partial}{\partial t}U_R\right)|\psi\rangle + U_R H U_R^\dagger|\psi_R\rangle \qquad \text{(Equation 12)}$$

$$= \left(i\hbar\left(\frac{\partial}{\partial t}U_R\right)U_R^\dagger + U_R H U_R^\dagger\right)|\psi_R\rangle \qquad \text{(Equation 13)}$$

We can take

$$U_R = \exp\left(iH_0 t/\hbar\right) \qquad \text{(Equation 14)}$$

then

$$i\hbar\frac{\partial}{\partial t}|\psi_R\rangle = \left(-H_0 + U_R(H_0 + H_1)U_R^\dagger\right)|\psi_R\rangle \qquad \text{(Equation 15)}$$

$$= U_R H_1 U_R^\dagger|\psi_R\rangle \qquad \text{(Equation 16)}$$

$$= \cos\omega t \begin{pmatrix} H_{11} & H_{12}e^{i(E_{11}-E_{10})t/\hbar} & H_{13}e^{i(E_{11}-E_{01})t/\hbar} & H_{14}e^{i(E_{11}-E_{00})t/\hbar} \\ H_{21}e^{-i(E_{11}-E_{10})t/\hbar} & H_{22} & H_{23}e^{i(E_{10}-E_{01})t/\hbar} & H_{24}e^{i(E_{10}-E_{00})t/\hbar} \\ H_{31}e^{-i(E_{11}-E_{01})t/\hbar} & H_{32}e^{-i(E_{10}-E_{01})t/\hbar} & H_{33} & H_{34}e^{i(E_{01}-E_{00})t/\hbar} \\ H_{41}e^{-i(E_{11}-E_{00})t/\hbar} & H_{42}e^{-i(E_{10}-E_{00})t/\hbar} & H_{43}e^{-i(E_{01}-E_{00})t/\hbar} & H_{44} \end{pmatrix}|\psi_R\rangle$$

(Equation 17)

III. APPLYING AC PULSES TO SYSTEM

In our system, it is useful to set our two detunings ϵ_1 and ϵ_2 to zero. When this is the case then our unperturbed Hamiltonian becomes

$$H_0 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 \\ 0 & 0 & 0 & -\lambda_1 \end{pmatrix} \qquad \text{(Equation 18)}$$

where

$$\lambda_1 = \sqrt{(\Delta_1 + \Delta_2)^2 + g^2}, \qquad \lambda_2 = \sqrt{(\Delta_1 - \Delta_2)^2 + g^2} \qquad \text{(Equation 19)}$$

This makes the perturbation in the rotating frame (shown in (Equation 17)):

$$i\hbar \frac{\partial}{\partial t}|\psi_R\rangle = \cos\omega t \begin{pmatrix} H_{11} & H_{12}e^{i(\lambda_1-\lambda_2)t/\hbar} & H_{13}e^{i(\lambda_1+\lambda_2)t/\hbar} & H_{14}e^{i(2\lambda_1)t/\hbar} \\ H_{21}e^{-i(\lambda_1-\lambda_2)t/\hbar} & H_{22} & H_{23}e^{i(2\lambda_2)t/\hbar} & H_{24}e^{i(\lambda_1+\lambda_2)t/\hbar} \\ H_{31}e^{-i(\lambda_1+\lambda_2)t/\hbar} & H_{32}e^{-i(2\lambda_2)t/\hbar} & H_{33} & H_{34}e^{i(\lambda_1-\lambda_2)t/\hbar} \\ H_{41}e^{-i(2\lambda_1)t/\hbar} & H_{42}e^{-i(\lambda_1+\lambda_2)t/\hbar} & H_{43}e^{-i(\lambda_1-\lambda_2)t/\hbar} & H_{44} \end{pmatrix} |\psi_R\rangle$$

(Equation 20)

To see what these perturbative Hamiltonians will look like, let us consider pulsing the detuning of the first dot.

In the (LR) basis, this matrix takes the form

$$H_1^{\epsilon_1,\mathrm{(LR)}} = \frac{1}{2} \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & 0 & 0 & -B \end{pmatrix} \cos\omega t \qquad \text{(Equation 21)}$$

However, to use (Equation 20), we must first change into the energy basis. After applying the unitary transformation, we get:

$$H_1^{\epsilon_1} = \frac{1}{2}\begin{pmatrix} 0 & A_1 \times \text{Sign}(\Delta_1 - \Delta_2) & -A_2 & 0 \\ A_1 \times \text{Sign}(\Delta_1 - \Delta_2) & 0 & 0 & -A_2 \times \text{Sign}(\Delta_1 - \Delta_2) \\ -A_2 & 0 & 0 & -A_1 \\ 0 & -A_2 \times \text{Sign}(\Delta_1 - \Delta_2) & -A_1 & 0 \end{pmatrix} \cos(\omega t)$$

(Equation 22)

where A_1 and A_2 are constants, and functions of Δ_1, Δ_2, and g.

Which, in the rotating frame, becomes

$$i\hbar\frac{\partial}{\partial t}|\psi_R\rangle = \frac{1}{2}\cos\omega t \begin{pmatrix} 0 & A_1 e^{i(\lambda_1-\lambda_2)t/\hbar} & -A_2 e^{i(\lambda_1+\lambda_2)t/\hbar} & 0 \\ A_1 e^{-i(\lambda_1-\lambda_2)t/\hbar} & 0 & 0 & -A_2 e^{i(\lambda_1+\lambda_2)t/\hbar} \\ -A_2 e^{-i(\lambda_1+\lambda_2)t/\hbar} & 0 & 0 & -A_1 e^{i(\lambda_1-\lambda_2)t/\hbar} \\ 0 & -A_2 e^{-i(\lambda_1+\lambda_2)t/\hbar} & -A_1 e^{-i(\lambda_1-\lambda_2)t/\hbar} & 0 \end{pmatrix} |\psi_R\rangle$$

(Equation 23)

if $\Delta_1 > \Delta_2$, and

$$i\hbar\frac{\partial}{\partial t}|\psi_R\rangle = \frac{1}{2}\cos\omega t \begin{pmatrix} 0 & -A_1 e^{i(\lambda_1-\lambda_2)t/\hbar} & -A_2 e^{i(\lambda_1+\lambda_2)t/\hbar} & 0 \\ -A_1 e^{-i(\lambda_1-\lambda_2)t/\hbar} & 0 & 0 & A_2 e^{i(\lambda_1+\lambda_2)t/\hbar} \\ -A_2 e^{-i(\lambda_1+\lambda_2)t/\hbar} & 0 & 0 & -A_1 e^{i(\lambda_1-\lambda_2)t/\hbar} \\ 0 & A_2 e^{-i(\lambda_1+\lambda_2)t/\hbar} & -A_1 e^{-i(\lambda_1-\lambda_2)t/\hbar} & 0 \end{pmatrix} |\psi_R\rangle$$

(Equation 24)

if $\Delta_1 < \Delta_2$.

Let us say that $\Delta_1 > \Delta_2$, meaning that the Hamiltonian in the rotating frame looks like (Equation 23). If we pulse at a frequency $\omega = (\lambda_1 + \lambda_2)/\hbar$, then in the rotating wave approximation, the equation of motion becomes

$$i\hbar\frac{\partial}{\partial t}|\psi_R\rangle = \frac{1}{4}\begin{pmatrix} 0 & 0 & -A_2 & 0 \\ 0 & 0 & 0 & -A_2 \\ -A_2 & 0 & 0 & 0 \\ 0 & -A_2 & 0 & 0 \end{pmatrix}|\psi_R\rangle \qquad \text{(Equation 25)}$$

which corresponds to rotations around the X axis of the second qubit. The Rabi frequency associated with this is $A_2/\hbar$.

The wiki has a full list of all of the pulses, with all of the associated rotation matrices.

IV. LIST OF LOGICAL GATES

- Z_1 gate (Z on qubit 1)
  - Wait for a time $\tau = \frac{h}{2(\lambda_1+\lambda_2)}$ (always on)
- Z_2 gate (Z on qubit 2)
  - Wait for a time $\tau = \frac{h}{2(\lambda_1-\lambda_2)}$ (always on)
- X_1 gate (X on qubit 1)
  - Pulse ϵ_1 at a frequency of $\omega_{AC} = (\lambda_1 + \lambda_2)/\hbar$ for a time $\tau = \frac{2h}{A_2}$
- X_2 gate (X on qubit 2)
  - Pulse ϵ_2 at a frequency of $\omega_{AC} = (\lambda_1 - \lambda_2)/\hbar$ for a time $\tau = \frac{2h}{C_1}$
- CNOT_1 gate (CNOT with qubit 1 as control)
  - Pulse ϵ_1 at a frequency of $\omega_{AC} = (\lambda_1 - \lambda_2)/\hbar$ for a time $\tau = \frac{h}{A_1}$
  - Pulse ϵ_2 at a frequency of $\omega_{AC} = (\lambda_1 - \lambda_2)/\hbar$ for a time $\tau = \frac{h}{C_1}$
- CNOT_2 gate (CNOT with qubit 2 as control)
  - Pulse ϵ_2 at a frequency of $\omega_{AC} = (\lambda_1 + \lambda_2)/\hbar$ for a time $\tau = \frac{h}{C_2}$

  - Pulse ϵ_1 at a frequency of $\omega_{AC} = (\lambda_1 + \lambda_2)/\hbar$ for a time $\tau = \frac{3h}{A_2}$

- SWAP
  - Pulse either Δ_1 or Δ_2 at a frequency of $\omega_{AC} = 2\lambda_2/\hbar$ for a time $\tau = \frac{2h\lambda_2}{Bg}$