finite, and thus $dq/d\phi = \infty$ at such points. This cusp has a natural physical interpretation. In order to minimize the time of transit, the particle must gain speed as quickly as possible. This can only be done by moving along a perpendicular to the equipotential, i.e., along the local field line (along the gravitational field or the fictitious centrifugal field) at the initial point where the particle starts from rest. This yields the characteristic cusp at these points.

ACKNOWLEDGMENTS

The author would like to thank V. J. Borowski, Ford Motor Company, and Paul Weiss, Wayne State University, for a number of stimulating and helpful discussions.

APPENDIX

Let $u^2 = r^2 - c^2 \geqslant 0$; then (27) can be written

$$\left(\frac{ds}{dr}\right)^2 = \frac{r^2}{u^2 + \lambda^2 s^2} \geqslant 1, \tag{A1}$$

where $\lambda^2 = mc^2 \omega_0^2/2E$. Since r increases from c as s increases from 0, we have

$$ds = u\, du(u^2 + \lambda^2 s^2)^{-1/2}, \quad s > 0. \tag{A2}$$

Since (A2) is positive homogeneous of degree 1 in u and s (i.e., it is invariant to the simultudinous transformation group $s \to ks$, $u \to ku$), we let

$$z = \lambda s/(u^2 + \lambda^2 s^2)^{1/2}, \tag{A3}$$

and find that the differential equation (A2) in s and z separates, i.e.,

$$\frac{dz}{z(1 - \lambda^{-1} z - z^2)} = \frac{ds}{s}.$$

Integration of (A3) yields the general solution to (27)

$$\frac{1}{2} \ln \frac{z^2}{|\lambda - z - \lambda z^2|} + \frac{1}{2} \frac{1}{(4\lambda^2 + 1)^{1/2}} \times \ln \frac{|2\lambda z + 1 + (4\lambda^2 + 1)^{1/2}|}{|2\lambda z + 1 - (4\lambda^2 + 1)^{1/2}|} = \ln s + C, \tag{A4}$$

$z > 0$. The special solution

$$s^2 = A(r^2 - c^2)$$

corresponds to the case

$$z = \text{constant} = [-1 + (4\lambda^2 + 1)^{1/2}]/2\lambda.$$

[2]P. W. Cooper, Am. J. Phys. **34**, 68 (1966).
[2]G. Venezian, Am. J. Phys. **34**, 701 (1966).
[3]R. L. Mallet, Am. J. Phys. **34**, 702 (1966).
[4]L. J. Laslett, Am. J. Phys. **34**, 702 (1966).
[5]J. Patel, Am. J. Phys. **35**, 436 (1967).
[6]S. M. Lee, Am. J. Phys. **40**, 315 (1972).
[7]J. R. Prussing, Am. J. Phys. **44**, 304 (1976).
[8]R. Chander, Am. J. Phys. **45**, 848 (1977).
[9]A. Gray and J. G. Gray, *A Treatise on Dynamics* (Macmillian, London, 1920), 2nd ed., p. 176.
[10]J. M. McKinley, Am. J. Phys. **47**, 81 (1979).
[11]E. J. Routh, *Dynamics of a System of Rigid Bodies, Part I* (Macmillan, London, 1905; reprinted by Dover, New York, 1960), 7th ed., pp. 408–413.
[12]A. Sommerfeld, *Mechanics* (Academic, New York, 1964), p. 95.
[13]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1980), 2nd ed., p. 43.
[14]H. H. Denman, J. Math. Phys. **6**, 1611 (1965).
[15]H. H. Denman, J. Math. Phys. **7**, 1910 (1966).
[16]A. Cohen, *An Introduction to Lie Theory of One-Parameter Groups* (Stechert, New York, 1931); also, G. W. Bluman and J. D. Cole, *Similarity Methods for Differential Equations* (Springer-Verlag, Berlin, 1974).
[17]H. H. Denman and L. H. Buch, SIAM J. Appl. Math. **40**, 279 (1981).
[18]D. G. Currie and E. J. Saletan, J. Math. Phys. **7**, 967 (1966).
[19]E. L. Ince, *Ordinary Differential Equations* (Longmans, Green, and Co., London, 1927; reprinted by Dover, New York, 1956).
[20]V. J. Borowski and H. H. Denman, Patent application (1979), Ford Motor Company.
[21]L. Landau and E. Lifshitz, *Mechanics* (Addison-Wesley, Reading, MA, 1960), p. 27.
[22]However, the converse (i.e., does a symmetry of the Euler–Lagrange equation imply the symmetry in a Lagrangian generating it?) is not true in general. See Refs. 12 and 13. For a more general discussion of symmetry in physical systems, see, e.g., J. Rosen, *Symmetry Discovered* (Cambridge University, Cambridge, 1975) and J. Rosen and Y. Freundlich, Am. J. Phys. **46**, 1030 (1978) on symmetries and conservation laws.

Hole gratings for optics experiments

T. A. Wiggins
Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802

(Received 20 February 1984; accepted for publication 4 April 1984)

The use of hole gratings, a regular arrangement of round holes in a thin opaque sheet, in an advanced laboratory is discussed. The effects of the hole size, spacing, and arrangement and of the grating aperture are demonstrated. Experiments are suggested which illustrate interference and diffraction effects and spatial filtering.

I. INTRODUCTION

A hole grating is a regular arrangement of equal-sized holes, usually round, in an otherwise opaque sheet. Hole gratings have been used to sample high-energy laser beams and for the production of a set of images of a source with a wide range of intensities.[1,2] They may be used in transmission if the sheet is thin and if it will survive the incident

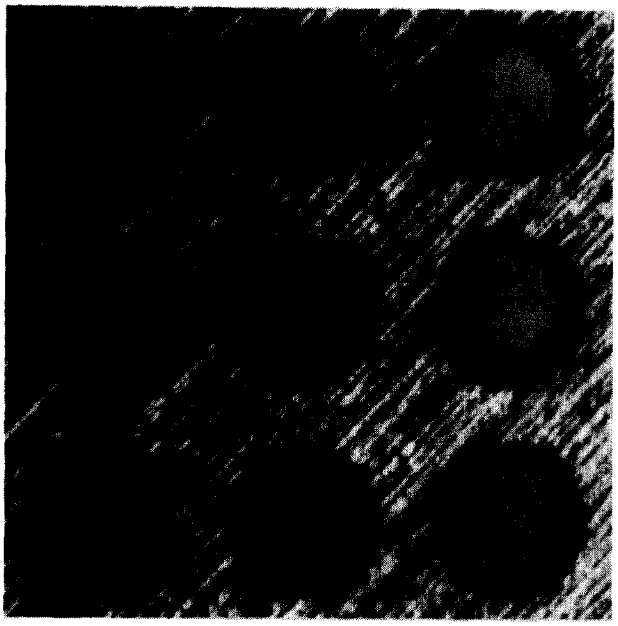

Fig. 1. Detail of a hole grating. The sloping shoulders around each hole appear black.

intensity. In some applications they have been used in reflection.

Hole gratings have also been found useful in upper-level optics courses as examples for interference and diffraction experiments. The gratings used were pieces of screens produced by Buckbee-Mears Co. using photo-etching of stainless steel sheets. Several hole spacings, hole sizes, and hole arrangements were available.[3] They are very uniform and have withstood repeated irradiations from pulsed Nd:YAG (yttrium aluminum garnet) and ruby laser systems. The detail in one of these gratings is shown in Fig. 1. They should be kept clean and flat. Mounting between glass plates is not recommended due to arbitrary phase shifts produced by the glass.

II. EXPERIMENTAL ARRANGEMENT

For use in a laboratory exercise, gratings are illuminated with a smooth parallel beam. This beam can best be produced by using a spatial filter, i.e., a short focal length lens such as a microscope objective and a pinhole, in front of a small laser followed by a lens of focal length f_1 placed so that the pinhole is at its focal point. The positioning of this lens can be accomplished by autocollimation. The focal length f_1 is selected using one of the two following criteria which represent extreme conditions: (1) long enough so that the Airy disk due to the pinhole has a large diameter producing a sensibly uniform intensity over the aperture of the grating, or (2) short enough so that several rings in the pinhole's diffraction pattern are within the aperture of the grating to be used. The hole grating is placed anywhere in this collimated beam. A second lens of focal length f_2 is used to produce a pattern in its focal plane which can be photographed for quantitative measurements or viewed with an eyepiece for qualitative observations. The distance between the two lenses is not important but placing the hole grating at the second lens's front focal point will be required for one of the experiments noted below. Both lenses should have apertures larger than that of the hole grating. For a pinhole of about 30 μm, f_1 should be 50 cm or larger in the first case or about 20 cm for the second case assuming a grating aperture of 2 cm. A value for f_2 of about 1 m is convenient. Adequate pinholes can be made with a fine needle piercing aluminum foil placed on a smooth, soft surface. The size and shape of the pinhole can be determined from its diffraction pattern.

III. OBSERVATIONS

Using the larger value of f_1, observations can be made of four properties of a hole grating. These are hole size, hole spacing, hole arrangement, and grating aperture. In addition, if lasers producing different wavelengths are available, the effects of changing the wavelength can be studied.

First, the production of a set of spots can be analyzed. Elementary considerations using small angle approximations show that constructive interference will occur when $D = m\lambda f_2/d$. D is the distance in the image plane between the central spot and any other spot along one of two mutually perpendicular directions, λ is the incident wavelength, d is the separation of the holes in the grating, and m is any integer. For a two-dimensional array, m is replaced by $(m^2 + n^2)^{1/2}$, where m and n indicate the order of interference along perpendicular directions. The effect of changing λ, d, and f_2 can be observed for the several spots which can be designated by m and n along perpendicular directions.

Second, the Airy disk corresponding to a single hole can be seen. The radius of this disk can be predicted from $A = 0.61\lambda f_2/a$, where a is the radius of a hole. Again, the parameters λ, f_2, and a can be used. Further, the order (m,n) of the spot which would occur at the position of the first zero in the Airy pattern can be predicted from $(m,n) = 0.61d/a$, a result independent of f_2 and λ. Higher-order zeros in the Airy disk may also be seen.

Third, the hole arrangement can be deduced from the appearance of the pattern. A square array of holes produces a square array in the pattern. Tipping the hole grating about an axis parallel to a row of holes will produce a rectangular pattern with the Airy disk of a round hole becoming elliptical. Extreme tipping is not recommended due to the finite thickness of the sheet. Arrays having 60° rather than 90° symmetry can also be studied. For the latter case, the distance D between spots is calculated using $2D = (m^2 + 3n^2)^{1/2} \lambda f_2/d$, where $m + n$ is an even integer. Spots are numbered $m = 0,2,4,...$ along a direction of major symmetry, and $n = 0,1,2...$ for rows parallel to this direction. Analysis can also be done with a nonrectangular coordinate system.

Fourth, the effect of the aperture of the hole grating can be studied. Placing a limiting aperture in the collimated beam causes structure to appear around each of the spots formed by the second lens. Using an adjustable iris, each spot is the center of an Airy pattern whose dimensions are inversely proportional to the size of the iris. For a rectangular aperture, maxima and minima appear whose spacing is inversely proportional to the aperture dimensions. The intensity distribution can be determined from $T_D = T(\sin\alpha/\alpha)^2$, where $\alpha = \pi aD/\lambda f_2$, D being the distance from the center of a spot of intensity T, and a is the aperture of the grating. If a rectangular aperture whose sides are parallel to the rows of holes is made small enough so that only about a dozen holes along a straight line on the grating are illuminated, the $N-2$ maxima between spots can be observed, where N is the number of holes illuminated.

Figures 2 and 3 show typical results which can be pro-

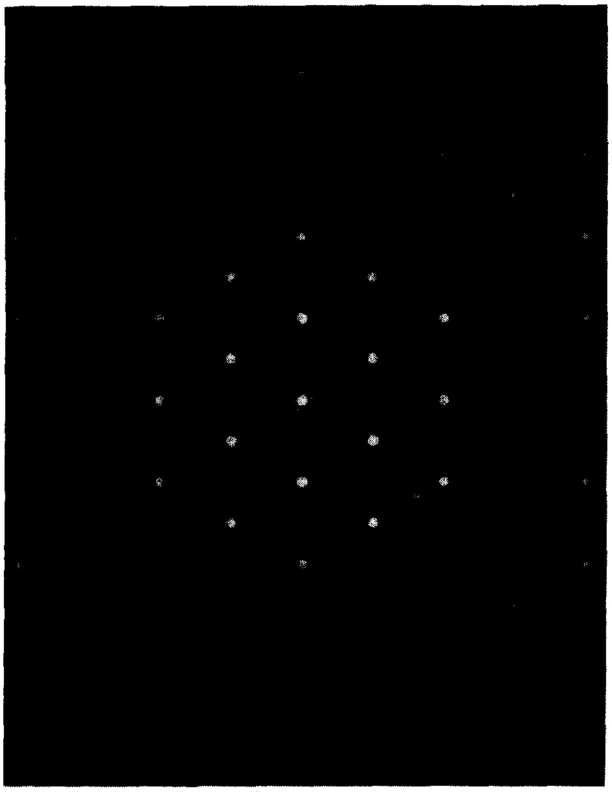

Fig. 2. Interference-diffraction pattern produced by a 60° array of round holes using a small circular aperture. Note that rotation of the figure by 60° reproduces the pattern shown.

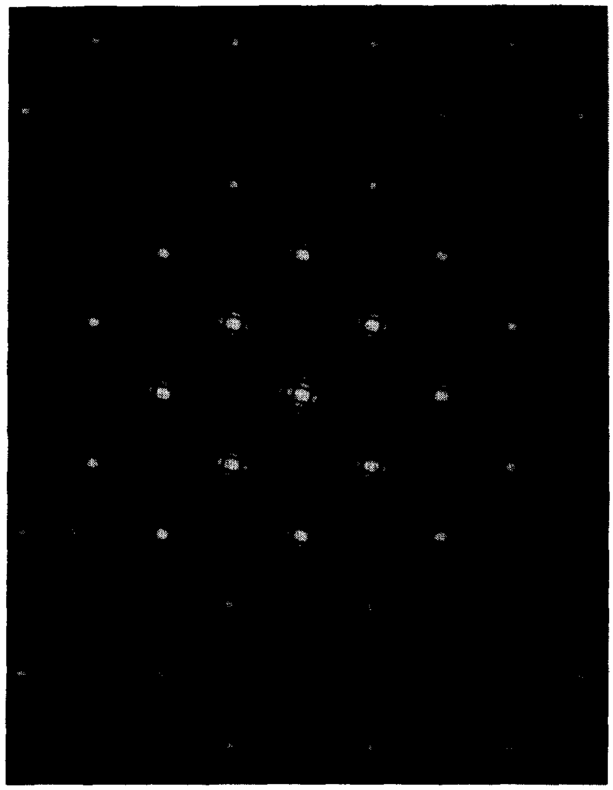

Fig. 3. Interference-diffraction pattern produced by a 90° array of round holes using a rectangular aperture whose sides are not parallel to the axes of the array. Note that rotation of the figure by 90° reproduces the positions of the intense regions of the pattern shown.

duced using these hole gratings. Figure 2 shows the pattern of a 60° hole grating having a small circular aperture. The spacing of the spots, the Airy disk due to a single hole, the 60° symmetry of the spots, and the Airy pattern around each spot are evident. Figure 3 shows the result with a 90° hole grating having a rectangular aperture whose sides are not aligned with the rows of holes in the grating. Again, hole spacing, size, arrangement, and aperture can be studied.

Three additional observations can be made using hole gratings. For these, the first lens should have a small focal length so that there is effectively no limiting aperture due to the grating or lenses.

First, the size of the pinhole can be determined from the size of any spot and the ratio f_2/f_1. This is essentially a geometrical optics experiment where the role of the hole grating is simply to produce multiple images of the pinhole with a wide intensity range.

Second, the transmission T of the grating and the intensity distribution in the focal plane of the second lens may be studied if a suitable detector is available. Comparison of the total light in this focal plane with the grating in place compared with that with the grating removed gives the grating's transmission which can be correlated with the ratio a/d. This ratio is known from previous measurements or from measurement of the grating in a calibrated microscope. In addition, the intensity of the central spot alone compared with the intensity at the focus without the grating should be equal to the square of the transmission.[4] This quadratic dependence upon transmission is due to the intensity reductions by first, the opaque areas of the grating; and second, the transmitted energy goes into the production of many spots rather than the single spot formed in the absence of the grating. Further, the relative intensities of the several spots can be predicted from the ratio a/d. This analysis requires the use of tables of first-order Bessel functions, J_1, the relative intensity of a spot being proportional to the square of $J_1(\phi)/(\phi)$, where the argument of the function, $\phi = \pi a \sin\theta/\lambda$, depends upon the order of interference $(m,n) = d \sin\theta/\lambda$.

Third, the hole gratings may be used in an experiment in spatial filtering. For this work the grating should be placed at the front focal plane of the second lens. A third lens of focal length f_3 is placed a distance f_3 beyond the back focal plane of the second lens. A value for f_3 of 50 cm or more is convenient. This lens will form in its focal plane an image of the hole grating which can be studied with the use of an eyepiece. Its appearance will depend upon what apertures are placed at the common focal plane between the second and third lenses. Slits, wires, irises, and circular obstacles here will markedly affect its appearance and can lead to an exercise in Fourier transforms.

It is noted that if a contact print of a hole grating is made on high-contrast film, Babinet's principle can be explored.

[1]J. E. Harvey and M. L. Scott, Opt. Eng. **20**, 88 (1981).

[2]T. A. Wiggins, T. T. Saito, and R. M. Herman, Appl. Opt. **22**, 3388 (1983).

[3]A sampling of several screens can be obtained by sending a check for $10.00 to Micro Products Division, Attn: Maxine Pfeiffer, Customer Service Representative, Buckbee-Mears Co., 245 East 6th Street, St. Paul, MN 55101.

[4]Reference 1 and M. L. Scott (private communication).